A Converse Lyapunov–Krasovskii Theorem for the Global Asymptotic Local Exponential Stability of Nonlinear Time–Delay Systems

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Abstract— **In this paper, the notion of GALES (Global Asymptotic Local Exponential Stability) is extended to nonlinear systems described by Retarded Functional Differential Equations. Necessary and sufficient Lyapunov–Krasovskii conditions ensuring the GALES of nonlinear systems with state–delays are provided. The conditions related to the lower bound and to the dissipation rate of the Lyapunov–Krasovskii functional involve only the current value of the solution, making the provided tool easy to use. An example, concerning a glucose–insulin regulatory system, is presented.**

I. Introduction

The Lyapunov–Krasovskii approach for the stability analysis of nonlinear time–delay systems is often a common choice in the control systems community. Many converse Lyapunov theorems, for various global/local stability notions of systems described by retarded functional differential equations (RFDEs), can be found in the literature (see, among the others, [1], [2], [7], [8], [9], [10], [11], [12], [19], [21]). Differently from the delay–free case, the stability analysis of nonlinear systems with state delays, by the use of the Lyapunov–Krasovskii approach, involves the study of a functional which takes as argument the infinite dimensional state of the system under consideration. This fact makes such analysis harder with respect to the delay–free counterpart. Concerning the global asymptotic stability (GAS) of nonlinear systems with state–delays, it is well–known that the sufficient conditions related to the lower bound and to the dissipation rate of the Lyapunov–Krasovskii functional involve only the current value of the solution (see, for instance, [5], [14]). The simplicity of these conditions turns out to be very helpful in practical applications. On the other hand, if either the local exponential stability (LES) or the global exponential stability (GES) of nonlinear systems with state–delays is concerned, the only results provided in the literature require dissipation rate of the Lyapunov–Krasovskii functional involving the whole state (see, for instance, Theorem 1.3 in [14] and Lemma B.2 in [19]). Relaxed conditions for the GES property, where the dissipation rate of the Lyapunov–Krasovskii functional involves the solution only at the current time instant, are provided in [1] for nonlinear globally Lipschitz time–delay systems. We highlight also that, to our best knowledge, in the literature concerning nonlinear systems with state delays, the Lyapunov characterization of contemporary global asymptotic and local exponential stability (GALES) has never been studied. The GALES notion has been very recently introduced in [6] for nonlinear delay–free systems and we believe it will be very significant in future developments of nonlinear control theory. For instance, it has been proved in [6] to be very useful in the stability analysis of nonlinear sampled–data control systems.

Motivated by these reasons, in the present paper, the GALES notion is extended to nonlinear systems with state–delays. In particular, it is proved that the existence of a Lyapunov–Krasovskii functional, satisfying suitable conditions, is necessary and sufficient for the GALES property of nonlinear time–delay systems. These conditions are easy to use and, furthermore, the globally Lipschitz property of the function describing the system (see [1]) is not needed here since only the local exponential stability property is concerned. On the other hand, the same conditions still ensure the global asymptotic stability property. An example, concerning a glucose–insulin regulatory system is presented to show the ease of use of the provided tool. In particular, it is proved that a suitable static state feedback controller, ensuring the semi–global practical stability when applied in discrete–time basis (see [3]), yields the GALES property of the related closed–loop system when applied in continuous–time basis.

Notation $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^+$ denotes the set of nonnegative reals $[0,+\infty)$. The symbol $|\cdot|$ stands for the Euclidean norm of a real vector, or the induced Euclidean norm of a matrix. For a given positive integer n, for a symmetric, positive definite matrix $P \in \mathbb{R}^{n\times n}$, $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote the maximum and the minimum eigenvalue of P, respectively. The essential supremum norm of an essentially bounded function is indicated with the symbol $\|\cdot\|_\infty$. For a positive integer n, for a nonnegative real Δ (maximum involved time–delay): $\mathcal{C}^n$ denotes the space of the continuous functions mapping $[-\Delta,0]$ into $\mathbb{R}^n$; $\mathcal{C}^{1,n}$ denotes the space of the continuously differentiable functions mapping $[-\Delta,0]$ into $\mathbb{R}^n$. For a positive real p, for $\phi \in \mathcal{C}^n$, $\mathcal{C}^n_p(\phi) = \{\psi \in \mathcal{C}^n : \|\psi-\phi\|_\infty \leq p\}$. The symbol $\mathcal{C}^n_p$ denotes $\mathcal{C}^n_p(0)$. For a continuous function $x : [-\Delta, c) \to \mathbb{R}^n$, with $0 < c \leq +\infty$, for any real $t \in [0,c)$, x_t is the function in $\mathcal{C}^n$ defined as $x_t(\tau) = x(t+\tau)$, $\tau \in [-\Delta,0]$. Let us

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here recall that a continuous function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is: of class $\mathcal{P}_0$ if $\gamma(0) = 0$; of class $\mathcal{P}$ if it is of class $\mathcal{P}_0$ and $\gamma(s) > 0$, $s > 0$; of class $\mathcal{K}$ if it is of class $\mathcal{P}$ and strictly increasing; of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded; of class $\mathcal{L}$ if it strictly decreases to zero as its argument tends to $+\infty$. A continuous function $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is of class $\mathcal{KL}$ if, for each fixed $t \geq 0$, the function $s \to \beta(s,t)$ is of class $\mathcal{K}$ and, for each fixed $s \geq 0$, the function $t \to \beta(s,t)$ is of class $\mathcal{L}$.

II. Preliminaries

Let us consider a nonlinear time–delay system described by the following RFDE (see [5], [14])

$$\begin{aligned} \dot{x}(t) &= f(x_t), \quad t \geq 0, \\ x(\tau) &= x_0(\tau), \quad \tau \in [-\Delta, 0], \end{aligned} \tag{1}$$

where: $x(t) \in \mathbb{R}^n$, n is a positive integer; x_0, $x_t \in \mathcal{C}^n$; $\Delta > 0$ is the maximum involved time delay; f is a function from $\mathcal{C}^n$ to $\mathbb{R}^n$, Lipschitz on bounded subsets of $\mathcal{C}^n$. It is assumed that $f(0) = 0$. The following definition recalls the GALES notion, introduced in [6], in the formulation for systems described by (1).

Definition 1: The system described by (1) is said to be 0–GALES if it is 0–GAS (see Definition 2.1 in [19]) and 0–LES (see Definition 2.1 in [15]).

III. Converse Theorem for the GALES Property

The following lemma, which characterizes the 0–GALES property by means of comparison functions (see [6]), is needed for the proof of the main result of the paper. The same reasoning in the proof of Proposition A.1 in [6], concerning the delay–free case, is here used, by exploiting Theorem 2.2 in [9] (see, also, Lemma A.1 in [19]).

Lemma 1: The system described by (1) is 0–GALES if and only if there exist a monotone non–decreasing, continuous function $\beta : \mathbb{R}^+ \to \mathbb{R}^+$, satisfying $\beta(0) > 0$, and a positive real λ such that, for any initial condition $x_0 \in \mathcal{C}^n$, the corresponding solution of (1) exists $\forall t \geq 0$ and, furthermore, satisfies the inequality

$$\|x_t\|_\infty \leq \beta(\|x_0\|_\infty)\|x_0\|_\infty e^{-\lambda t}, \quad t \geq 0. \tag{2}$$

Proof: The sufficiency part of the lemma is obvious. Let us prove the necessity part. By the 0–GAS property of the system described by (1), it follows that there exists a function $\bar{\beta}$ of class $\mathcal{KL}$ such that, $\forall x_0 \in \mathcal{C}^n$, (see Lemma A.1 in [19])

$$\|x_t\|_\infty \leq \bar{\beta}(\|x_0\|_\infty, t), \quad t \in [0, +\infty). \tag{3}$$

From (3), by the use of Lemma 7 and 8 in [13], we can conclude that there exist a function γ of class $\mathcal{K}_\infty$ and a function σ of class $\mathcal{L}$ such that, for any $x_0 \in \mathcal{C}^n$, the following condition holds (see inequality after (9.4) in [6]):

$$\|x_t\|_\infty \leq \bar{\beta}(\|x_0\|_\infty, t) \leq \gamma(\|x_0\|_\infty)\sigma(t), \quad t \in [0, +\infty). \tag{4}$$

By the 0–LES property of the system described by (1), for some positive real $R > 0$, there exist positive reals M and λ such that, for any $x_0 \in \mathcal{C}^n_R$, the following inequality holds:

$$\|x_t\|_\infty \leq M\|x_0\|_\infty e^{-\lambda t}, \quad t \in [0, +\infty). \tag{5}$$

On the other hand, in the case $\|x_0\|_\infty > R$, from (3), it follows that there exists a finite time $T(\|x_0\|_\infty) > 0$ such that $\|x_t\|_\infty \leq R$, $\forall t \geq T(\|x_0\|_\infty)$. By choosing, for instance, $T(\|x_0\|_\infty) = \sigma^{-1}(R/\gamma(\|x_0\|_\infty))$, from (4), it follows that $x_t \in \mathcal{C}^n_R$, $\forall t \geq T(\|x_0\|_\infty)$. Then, from (5), $\forall t \geq T(\|x_0\|_\infty)$ the following inequalities hold:

$$\begin{aligned} \|x_t\|_\infty &\leq M\|x_{T(\|x_0\|_\infty)}\|_\infty e^{-\lambda(t - T(\|x_0\|_\infty))} \\ &\leq MRe^{-\lambda(t-T(\|x_0\|_\infty))} \leq M\|x_0\|_\infty e^{\lambda T(\|x_0\|_\infty)} e^{-\lambda t}. \end{aligned} \tag{6}$$

From (4), in the case $\|x_0\|_\infty > R$, $\forall t < T(\|x_0\|_\infty)$, the following inequalities hold:

$$\begin{aligned} \|x_t\|_\infty &\leq \gamma(\|x_0\|_\infty)\sigma(t) \leq \gamma(\|x_0\|_\infty)\sigma(0) \\ &\leq \frac{\|x_0\|_\infty}{R}\gamma(\|x_0\|_\infty)\sigma(0)e^{\lambda(T(\|x_0\|_\infty)-t)}. \end{aligned} \tag{7}$$

Let $\beta : \mathbb{R}^+ \to \mathbb{R}^+$ be the function defined for any $s \in \mathbb{R}^+$ as follows

$$\beta(s) = \max\left\{M, Me^{\lambda T(s)}, \frac{\gamma(s)}{R}\sigma(0)e^{\lambda T(s)}\right\}. \tag{8}$$

Taking into account (5), (6), (7), we can conclude that, for any $x_0 \in \mathcal{C}^n$, inequality (2) holds $\forall t \geq 0$, with the function β provided in (8). The proof of the lemma is complete. ■

The main result of the paper is given in the following theorem, which provides necessary and sufficient Lyapunov–Krasovskii conditions for the 0–GALES of the system (1).

Theorem 1: The system described by (1) is 0–GALES if and only if there exist a continuous functional $V : \mathcal{C}^n \to \mathbb{R}^+$, positive reals α_1, α_3 and a monotone non–decreasing continuous function $\alpha_2 : \mathbb{R}^+ \to \mathbb{R}^+$, with $\alpha_2(0) > 0$, such that:

1) for any $\phi \in \mathcal{C}^{1,n}$ the following conditions hold (see [5])

$$\begin{aligned} \alpha_1|\phi(0)|^2 &\leq V(\phi) \leq \alpha_2(\|\phi\|_\infty)\|\phi\|^2_\infty, \\ D^+V(\phi) &\leq -\alpha_3|\phi(0)|^2, \end{aligned} \tag{9}$$

where D^+V denotes the upper–right Dini derivative of V along the solutions of (1), defined as (see [5])

$$D^+V(\phi) := \limsup_{h \to 0^+} \frac{V(x_h(\phi)) - V(\phi)}{h}; \tag{10}$$

2) the function $t \to V(x_t(\phi))$ is locally absolutely continuous in $[0, b)$, $0 < b \leq +\infty$, where $[0, b)$ is the maximal interval of existence of the solution $x(t)$ with initial condition $\phi \in \mathcal{C}^{1,n}$.

Proof: Sufficiency. In the following, we will prove that, under the conditions 1) and 2), the system (1) is 0–GAS and 0–LES. Without any loss of generality, it is assumed that the initial state $x_0 \in \mathcal{C}^{1,n}$ (see Proposition 3 in [20]). The 0–GAS property of the system described by (1) can be proved following the same steps of the proof of Theorem 2.1 in [5]

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(see Chapter 5, Theorem 2.1 in [5]) with the functions u, v, w defined, for any $s \in \mathbb{R}^+$, as $u(s) = \alpha_1 s^2$, $v(s) = \alpha_2(s)s^2$ and $w(s) = \alpha_3 s^2$, respectively. Concerning the 0–LES property of the system described by (1), by the same reasoning used in [6], from the first two inequalities in (9), for some positive real $E > 0$, there exists a positive real $l_1 \geq \alpha_1 > 0$ such that, for any $\phi \in \mathcal{C}^n_E$, the following condition holds: $V(\phi) \leq l_1 \|\phi\|^2_\infty$. As in [6], let $R = E\left(\alpha_1/l_1\right)^{\frac{1}{2}} \leq E$ and assume the initial condition $x_0 \in \mathcal{C}^n_R \cap \mathcal{C}^{1,n}$. As already noted, $x_0 \in \mathcal{C}^{1,n}$ does not imply any loss of generality (see Proposition 3 in [20]). From (9), it follows that $x_t(x_0) \in \mathcal{C}^n_E$, $\forall t \geq 0$, where $x_t(x_0)$ is the solution of system (1) with initial condition $x_0 \in \mathcal{C}^n_R \cap \mathcal{C}^{1,n}$. Then, from 1) and 2), it follows that, for any $x_0 \in \mathcal{C}^n_R \cap \mathcal{C}^{1,n}$, $V(x_t(x_0)) \leq l_1 \|x_t(x_0)\|^2_\infty$, $\forall t \geq 0$. Let x_t be any the solution of the system described by (1) with initial condition $x_0 \in \mathcal{C}^n_R \cap \mathcal{C}^{1,n}$. Taking into account the Lipschitz property of the function f in (1), the absolute continuity property of the functional V (see condition 2)) and that the solution $x_t \in \mathcal{C}^n_E$, $\forall t \geq 0$, the same steps used in the proof of Theorem 1 in [1] can be repeated here for proving the 0–LES property of the system described by (1). The sufficiency part is proved.

Necessity. Let $V : \mathcal{C}^n \to \mathbb{R}^+$ be the functional defined, for any $\phi \in \mathcal{C}^n$, as

$$V(\phi) = \int_0^{+\infty} \|x_t(\phi)\|^2_\infty dt + \sup_{t\in[0,+\infty)} \|x_t(\phi)\|^2_\infty, \quad (11)$$

where $x_t(\phi)$ is the solution of the system described by (1) with initial condition ϕ. Taking into account the 0–GALES property of the system (1), let β and λ be the function and the positive real in (2), respectively (see Lemma 1).

Firstly, we prove the continuity property of the functional V in (11). In particular, we prove the following statement: given any $\phi \in \mathcal{C}^n$ and any $\epsilon > 0$, there exists $\delta > 0$ with $\delta < \epsilon$ such that, for any $\psi \in \mathcal{C}^n_\delta(\phi)$, the following inequality holds

$$|V(\psi) - V(\phi)| < \epsilon. \quad (12)$$

Let $\phi \in \mathcal{C}^n$, $\epsilon > 0$ be given. Taking into account (11), we have, for any $\psi \in \mathcal{C}^n$,

$$|V(\psi) - V(\phi)| \leq \left| \int_0^{+\infty} \|x_t(\psi)\|^2_\infty dt - \int_0^{+\infty} \|x_t(\phi)\|^2_\infty dt \right| + \left| \sup_{t\in[0,+\infty)} \|x_t(\psi)\|^2_\infty - \sup_{t\in[0,+\infty)} \|x_t(\phi)\|^2_\infty \right|. \quad (13)$$

Let $L_1 : \mathbb{R}^+ \to \mathbb{R}^+$ be the function defined, for any $s \in \mathbb{R}^+$, as follows $L_1(s) = \beta(s+\epsilon)(s+\epsilon) + \beta(s)s$. Taking into account the function $L_1(s)$, Lemma 1 and the continuity property of the solution with respect to the initial state (see Theorem 2.1 in [14]), let $\delta < \epsilon$ and T be positive reals such that the following conditions hold:

$$\max\left\{ \left(\beta^2(\|\phi\|_\infty + \epsilon)(\|\phi\|_\infty + \epsilon)^2 + \beta^2(\|\phi\|_\infty)\|\phi\|^2_\infty\right) \times \frac{e^{-2\lambda T}}{2\lambda},\ L_1^2(\|\phi\|_\infty)e^{-\lambda T} \right\} < \frac{\epsilon}{3}, \quad (14)$$

$$L_1(\|\phi\|_\infty)\|x_t(\psi) - x_t(\phi)\|_\infty < \min\left\{\frac{\epsilon}{3}, \frac{\epsilon}{3T}\right\}, \quad t \in [0,T], \quad \psi \in \mathcal{C}^n_\delta(\phi). \quad (15)$$

Let $\psi \in \mathcal{C}^n_\delta(\phi)$. Taking into account (14), (15), Lemma 1 and that $\|\psi\|_\infty \leq \|\phi\|_\infty + \epsilon$, the following inequalities hold:

$$\begin{aligned}
&\left| \int_0^{+\infty} \|x_t(\psi)\|^2_\infty dt - \int_0^{+\infty} \|x_t(\phi)\|^2_\infty dt \right| \\
&\leq \left| \int_0^{T} \|x_t(\psi)\|^2_\infty dt - \int_0^{T} \|x_t(\phi)\|^2_\infty dt \right| \\
&\quad + \left| \int_T^{+\infty} \|x_t(\psi)\|^2_\infty dt \right| + \left| \int_T^{+\infty} \|x_t(\phi)\|^2_\infty dt \right| \\
&\leq \int_0^{T} \left| (\|x_t(\psi)\|_\infty + \|x_t(\phi)\|_\infty)(\|x_t(\psi)\|_\infty - \|x_t(\phi)\|_\infty) \right| dt \\
&\quad + \left| \int_T^{+\infty} \|x_t(\psi)\|^2_\infty dt \right| + \left| \int_T^{+\infty} \|x_t(\phi)\|^2_\infty dt \right| \\
&\leq \int_0^{T} \Big(\beta(\|\psi\|_\infty)\|\psi\|_\infty + \beta(\|\phi\|_\infty)\|\phi\|_\infty\Big) \times \\
&\quad \times \|x_t(\psi) - x_t(\phi)\|_\infty dt + \int_T^{+\infty} \beta^2(\|\psi\|_\infty)\|\psi\|^2_\infty e^{-2\lambda t} dt \\
&\quad + \int_T^{+\infty} \beta^2(\|\phi\|_\infty)\|\phi\|^2_\infty e^{-2\lambda t} dt \\
&\leq \int_0^{T} L_1(\|\phi\|_\infty)\|x_t(\psi) - x_t(\phi)\|_\infty dt + \Big(\beta^2(\|\psi\|_\infty)\|\psi\|^2_\infty \\
&\quad + \beta^2(\|\phi\|_\infty)\|\phi\|^2_\infty\Big)\frac{e^{-2\lambda T}}{2\lambda} < \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2}{3}\epsilon.
\end{aligned} \quad (16)$$

Furthermore, the following inequalities hold:

$$\begin{aligned}
&\left| \sup_{t\in[0,+\infty)} \|x_t(\psi)\|^2_\infty - \sup_{t\in[0,+\infty)} \|x_t(\phi)\|^2_\infty \right| \\
&\leq \sup_{t\in[0,+\infty)} \left| \|x_t(\psi)\|^2_\infty - \|x_t(\phi)\|^2_\infty \right| \\
&\leq \sup_{t\in[0,+\infty)} \left| (\|x_t(\psi)\|_\infty + \|x_t(\phi)\|_\infty)(\|x_t(\psi)\|_\infty - \|x_t(\phi)\|_\infty) \right| \\
&\leq \sup_{t\in[0,+\infty)} \Big(\beta(\|\psi\|_\infty)\|\psi\|_\infty + \beta(\|\phi\|_\infty)\|\phi\|_\infty\Big) \times \\
&\qquad \times \|x_t(\psi) - x_t(\phi)\|_\infty \\
&\leq L_1(\|\phi\|_\infty) \max\left\{ \sup_{t\in[0,T]} \|x_t(\psi) - x_t(\phi)\|_\infty, \right. \\
&\qquad \left. \sup_{t\in[T,+\infty)} \|x_t(\psi) - x_t(\phi)\|_\infty \right\} \\
&\leq L_1(\|\phi\|_\infty) \max\left\{ \sup_{t\in[0,T]} \|x_t(\psi) - x_t(\phi)\|_\infty, \right. \\
&\qquad \left. \sup_{t\in[T,+\infty)} \|x_t(\psi)\|_\infty + \|x_t(\phi)\|_\infty \right\} \\
&\leq L_1(\|\phi\|_\infty) \max\left\{ \sup_{t\in[0,T]} \|x_t(\psi) - x_t(\phi)\|_\infty, \right. \\
&\qquad \left. L_1(\|\phi\|_\infty)e^{-\lambda T} \right\} < \frac{\epsilon}{3}.
\end{aligned} \quad (17)$$

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Then, from (13) and taking into account (16), (17), inequality (12) holds.

As far as the first two inequalities in (9) are concerned (see condition 1)), taking into account Lemma 1, they hold by choosing $\alpha_1 = 1$ and the function α_2 defined, for any $s \in \mathbb{R}^+$, as $\alpha_2(s) = H\beta^2(s)$, where $H = \left(1 + \frac{1}{2\lambda}\right)$. Indeed, taking into account (11), for any $\phi \in \mathcal{C}^n$, the following inequalities hold:

$$\begin{aligned} &V(\phi) \geq \sup_{t\in[0,+\infty)} \|x_t(\phi)\|_\infty^2 \geq \|\phi\|_\infty^2 \geq |\phi(0)|^2, \\ &V(\phi) \leq \int_0^{+\infty} \beta^2(\|\phi\|_\infty)\|\phi\|_\infty^2 e^{-2\lambda t} dt \\ &\quad + \sup_{t\in[0,+\infty)} \beta^2(\|\phi\|_\infty)\|\phi\|_\infty^2 e^{-2\lambda t} \leq H\beta^2(\|\phi\|_\infty)\|\phi\|_\infty^2. \end{aligned} \tag{18}$$

As far as the last inequality in (9) is concerned (see condition 1)), it holds by choosing $\alpha_3 = 1$. Indeed, taking into account (11), for any $\phi \in \mathcal{C}^n$, the following equalities/inequalities hold:

$$\begin{aligned} &D^+V(\phi) = \limsup_{h\to 0^+} \frac{V(x_h(\phi)) - V(\phi)}{h} \\ &= \limsup_{h\to 0^+} \frac{1}{h}\left(\int_0^{+\infty} \|x_t(x_h(\phi))\|_\infty^2 dt - \int_0^{+\infty} \|x_t(\phi)\|_\infty^2 dt \right. \\ &\quad \left. + \sup_{t\in[0,+\infty)} \|x_t(x_h(\phi))\|_\infty^2 - \sup_{t\in[0,+\infty)} \|x_t(\phi)\|_\infty^2 \right) \\ &\leq \limsup_{h\to 0^+} \frac{1}{h}\left(\int_0^{+\infty} \|x_{t+h}(\phi)\|_\infty^2 dt - \int_0^{+\infty} \|x_t(\phi)\|_\infty^2 dt \right. \\ &\quad \left. + \sup_{t\in[0,+\infty)} \|x_{t+h}(\phi)\|_\infty^2 - \sup_{t\in[0,+\infty)} \|x_t(\phi)\|_\infty^2 \right) \\ &\leq \limsup_{h\to 0^+} \frac{1}{h}\left(\int_h^{+\infty} \|x_t(\phi)\|_\infty^2 dt - \int_0^{+\infty} \|x_t(\phi)\|_\infty^2 dt \right. \\ &\quad \left. + \sup_{t\in[h,+\infty)} \|x_t(\phi)\|_\infty^2 - \sup_{t\in[0,+\infty)} \|x_t(\phi)\|_\infty^2 \right) \\ &\leq \limsup_{h\to 0^+} \frac{1}{h}\left(- \int_0^{h} \|x_t(\phi)\|_\infty^2 dt + \int_0^{+\infty} \|x_t(\phi)\|_\infty^2 dt \right. \\ &\quad \left. - \int_0^{+\infty} \|x_t(\phi)\|_\infty^2 dt \right) \leq -\|\phi\|_\infty^2 \leq -|\phi(0)|^2. \end{aligned} \tag{19}$$

As far as condition 2) is concerned, the local absolute continuity property of the functional V defined in (11) follows from the following statement, which we are going to prove: given any $\phi \in \mathcal{C}^{1,n}$ and any positive real T, there exists a positive real F such that, for any t_1, $t_2 \in [0,T]$, $t_1 \leq t_2$, the following condition holds

$$|V(x_{t_2}(\phi)) - V(x_{t_1}(\phi))| < F|t_2 - t_1|. \tag{20}$$

Let $\phi \in \mathcal{C}^{1,n}$ and $T > 0$ be given. Let t_1, $t_2 \in [0,T]$, satisfying $t_1 \leq t_2$. Let $L_2 : \mathbb{R}^+ \to \mathbb{R}^+$ be the function defined, for any $s \in \mathbb{R}^+$, as

$$L_2(s) = 2\beta\big(\beta(s)s\big)\beta(s)s. \tag{21}$$

Taking into account (11), we have that

$$\begin{aligned} &|V(x_{t_2}(\phi)) - V(x_{t_1}(\phi))| \\ &\leq \left| \int_0^{+\infty} \|x_\tau(x_{t_2}(\phi))\|_\infty^2 d\tau - \int_0^{+\infty} \|x_\tau(x_{t_1}(\phi))\|_\infty^2 d\tau \right| \\ &\quad + \left| \sup_{\tau\in[0,+\infty)} \|x_\tau(x_{t_2}(\phi))\|_\infty^2 - \sup_{\tau\in[0,+\infty)} \|x_\tau(x_{t_1}(\phi))\|_\infty^2 \right|. \end{aligned} \tag{22}$$

Notice that

$$\begin{aligned} &\left| \int_0^{+\infty} \|x_\tau(x_{t_2}(\phi))\|_\infty^2 d\tau - \int_0^{+\infty} \|x_\tau(x_{t_1}(\phi))\|_\infty^2 d\tau \right| \\ &\leq \int_0^{+\infty} \left| \|x_\tau(x_{t_2}(\phi))\|_\infty^2 - \|x_\tau(x_{t_1}(\phi))\|_\infty^2 \right| d\tau \\ &\leq \int_0^{+\infty} \Big| (\|x_\tau(x_{t_2}(\phi))\|_\infty + \|x_\tau(x_{t_1}(\phi))\|_\infty) \times \\ &\qquad \times (\|x_\tau(x_{t_2}(\phi))\|_\infty - \|x_\tau(x_{t_1}(\phi))\|_\infty) \Big| d\tau. \end{aligned} \tag{23}$$

Taking into account Lemma 1 and (21), for any $\tau \in [0,+\infty)$, the following inequalities hold

$$\begin{aligned} &\|x_\tau(x_{t_2}(\phi))\|_\infty + \|x_\tau(x_{t_1}(\phi))\|_\infty \\ &\leq \beta(\|x_{t_2}(\phi)\|_\infty)\|x_{t_2}(\phi)\|_\infty + \beta(\|x_{t_1}(\phi)\|_\infty)\|x_{t_1}(\phi)\|_\infty \\ &\leq 2\beta\big(\beta(\|\phi\|_\infty)\|\phi\|_\infty\big)\beta(\|\phi\|_\infty)\|\phi\|_\infty = L_2(\|\phi\|_\infty). \end{aligned} \tag{24}$$

Let $\Phi : [-\Delta,+\infty) \to \mathbb{R}^n$ be the function defined as

$$\Phi(s) = \begin{cases} \dfrac{d\phi(s)}{ds}, & s \in [-\Delta, 0], \\ f(x_s(\phi)), & s \in (0,+\infty), \end{cases} \tag{25}$$

where f is the function in (1). From (23), taking into account (24), (25), the following inequalities hold

$$\begin{aligned} &\left| \int_0^{+\infty} \|x_\tau(x_{t_2}(\phi))\|_\infty^2 d\tau - \int_0^{+\infty} \|x_\tau(x_{t_1}(\phi))\|_\infty^2 d\tau \right| \\ &\leq \int_0^{+\infty} L_2(\|\phi\|_\infty)\|x_\tau(x_{t_2}(\phi)) - x_\tau(x_{t_1}(\phi))\|_\infty d\tau \\ &\leq \int_0^{+\infty} L_2(\|\phi\|_\infty) \sup_{\theta\in[-\Delta,0]} |x(\tau + t_2 + \theta)(\phi) \\ &\qquad\qquad - x(\tau + t_1 + \theta)(\phi)| d\tau \\ &\leq \int_0^{+\infty} L_2(\|\phi\|_\infty) \sup_{\theta\in[-\Delta,0]} \left| \phi(-\Delta) + \int_{-\Delta}^{\tau+t_2+\theta} \Phi(s) ds \right. \\ &\qquad\qquad \left. - \phi(-\Delta) - \int_{-\Delta}^{\tau+t_1+\theta} \Phi(s) ds \right| d\tau \\ &\leq \int_0^{+\infty} L_2(\|\phi\|_\infty) \sup_{\theta\in[-\Delta,0]} \int_{\tau+t_1+\theta}^{\tau+t_2+\theta} |\Phi(s)| ds d\tau \\ &\leq L_2(\|\phi\|_\infty) \max\left\{ \int_0^{\Delta} \sup_{\theta\in[-\Delta,0]} \int_{\tau+t_1+\theta}^{\tau+t_2+\theta} |\Phi(s)| ds d\tau, \right. \\ &\qquad \left. \int_\Delta^{+\infty} \sup_{\theta\in[-\Delta,0]} \int_{\tau+t_1+\theta}^{\tau+t_2+\theta} |\Phi(s)| ds d\tau \right\}. \end{aligned} \tag{26}$$

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Let $L_e > 0$ be the positive real such that for any $s_1, s_2 \in \mathbb{R}^+$, the condition hold $|e^{-\lambda s_1} - e^{-\lambda s_1}| \le L_e|s_1 - s_2|$. Moreover, taking into account Lemma 1 and the Lipschitz property of the function f in (1), let $P = \sup_{\theta\in[-\Delta,\Delta+T]} |\Phi(\theta)|$ and L_f be the continuous non–decreasing function such that $|f(x_s(\phi))| \le L_f(\|\phi\|_\infty)\|x_s(\phi)\|_\infty$, $\forall s \in \mathbb{R}^+$ (see Lemma 4 in [19] and, also, [7], [10], [11]). From (26), the following inequalities hold

$$
\begin{aligned}
&\left| \int_0^{+\infty} \|x_\tau(x_{t_2}(\phi))\|_\infty^2 d\tau - \int_0^{+\infty} \|x_\tau(x_{t_1}(\phi))\|_\infty^2 d\tau \right| \\
&\le L_2(\|\phi\|_\infty) \max\Bigg\{ \Delta P(t_2 - t_1), \\
&\qquad \int_\Delta^{+\infty} \sup_{\theta\in[-\Delta,0]} \int_{\tau+t_1+\theta}^{\tau+t_2+\theta} L_f(\|\phi\|_\infty)\|x_s(\phi)\|_\infty \, ds\, d\tau \Bigg\} \\
&\le L_2(\|\phi\|_\infty) \max\Bigg\{ \Delta P(t_2 - t_1), \\
&\qquad \int_\Delta^{+\infty} \sup_{\theta\in[-\Delta,0]} \int_{\tau+t_1+\theta}^{\tau+t_2+\theta} L_f(\|\phi\|_\infty)\beta(\|\phi\|_\infty)\|\phi\|_\infty e^{-\lambda s} ds\, d\tau \Bigg\} \\
&\le L_2(\|\phi\|_\infty) \max\Bigg\{ \Delta P(t_2 - t_1), \int_\Delta^{+\infty} \sup_{\theta\in[-\Delta,0]} L_f(\|\phi\|_\infty) \times \\
&\qquad \times \beta(\|\phi\|_\infty)\|\phi\|_\infty \frac{e^{-\lambda\tau}}{\lambda}(e^{-\lambda(t_1+\theta)} - e^{-\lambda(t_2+\theta)}) d\tau \Bigg\} \\
&\le (t_2 - t_1) L_2(\|\phi\|_\infty) \max\Bigg\{ \Delta P,\ L_f(\|\phi\|_\infty)\beta(\|\phi\|_\infty) \times \\
&\qquad \times \|\phi\|_\infty \frac{e^{-\lambda\Delta}}{\lambda^2} L_e \Bigg\}.
\end{aligned} \tag{27}
$$

With the similar reasoning used to state (27), taking into account the positive reals P, L_e and the functions Φ, L_f defined before, the following inequalities hold

$$
\begin{aligned}
&\left| \sup_{\tau\in[0,+\infty)} \|x_\tau(x_{t_2}(\phi))\|_\infty^2 - \sup_{\tau\in[0,+\infty)} \|x_\tau(x_{t_1}(\phi))\|_\infty^2 \right| \\
&\le \sup_{\tau\in[0,+\infty)} L_2(\|\phi\|_\infty)\|x_\tau(x_{t_2}(\phi)) - x_\tau(x_{t_1}(\phi))\|_\infty \\
&\le L_2(\|\phi\|_\infty) \sup_{\tau\in[0,+\infty)} \sup_{\theta\in[-\Delta,0]} |x(\tau + t_2 + \theta)(\phi) \\
&\qquad - x(\tau + t_1 + \theta)(\phi)| \\
&\le L_2(\|\phi\|_\infty) \max\Bigg\{ \sup_{\tau\in[0,\Delta]} \sup_{\theta\in[-\Delta,0]} \int_{\tau+t_1+\theta}^{\tau+t_2+\theta} |\Phi(s)| ds, \\
&\qquad \sup_{\tau\in[\Delta,+\infty)} \sup_{\theta\in[-\Delta,0]} \int_{\tau+t_1+\theta}^{\tau+t_2+\theta} |\Phi(s)| ds \Bigg\} \\
&\le L_2(\|\phi\|_\infty) \max\Bigg\{ P(t_2 - t_1), \sup_{\tau\in[\Delta,+\infty)} L_f(\|\phi\|_\infty) \times \\
&\qquad \times \beta(\|\phi\|_\infty)\|\phi\|_\infty \frac{e^{-\lambda\tau}}{\lambda} L_e(t_2 - t_1) \Bigg\} \\
&\le (t_2 - t_1) L_2(\|\phi\|_\infty) \max\Bigg\{ P, \\
&\qquad L_f(\|\phi\|_\infty)\beta(\|\phi\|_\infty)\|\phi\|_\infty \frac{e^{-\lambda\Delta}}{\lambda} L_e \Bigg\}.
\end{aligned} \tag{28}
$$

From (22), taking into account (27), (28), by choosing

$$
F > 2L_2(\|\phi\|_\infty) \max\Bigg\{ \Delta P,\ P,\ L_f(\|\phi\|_\infty)\beta(\|\phi\|_\infty)\|\phi\|_\infty \times \frac{e^{-\lambda\Delta}}{\lambda} L_e,\ L_f(\|\phi\|_\infty)\beta(\|\phi\|_\infty)\|\phi\|_\infty \frac{e^{-\lambda\Delta}}{\lambda^2} L_e \Bigg\},
$$

inequality (20) holds. Then, condition 2) is proved. The proof of the theorem is complete. ■

IV. Example

Let us consider the glucose-insulin regulatory system described by the following RFDEs (see [16], [18] for more details on the model)

$$
\begin{aligned}
\dot{x}_1(t) &= -K_{xgi}\Big(x_1(t)x_2(t) + I_{\text{ref}}x_1(t) + G_{\text{ref}}x_2(t)\Big) \\
\dot{x}_2(t) &= -K_{xi}(x_2(t) + I_{\text{ref}}) \\
&\quad + \frac{T_{iG\max}}{V_I}\varphi\big(x_1(t-\tau_g) + G_{\text{ref}}\big) + \frac{v_{\text{ref}} + u(t)}{V_I} \\
x(\tau) &= x_0(\tau), \quad \tau \in [-\tau_g, 0]
\end{aligned} \tag{29}
$$

where $x_1(t)$, $x_2(t) \in \mathbb{R}$, $x_0 \in \mathcal{C}^2$, τ_g is the involved time delay, $u(t) \in \mathbb{R}$ is the input, I_{ref} and v_{ref} are defined, for a chosen G_{ref} (glucose desired level), as $I_{\text{ref}} = T_{gh}/(V_G G_{\text{ref}} K_{xgi})$, $v_{\text{ref}} = V_I I_{\text{ref}} K_{xi} - T_{iG\max}\varphi(G_{\text{ref}})$. Let $k : \mathcal{C}^2 \to \mathbb{R}$ be the function defined, for any $\phi = \begin{bmatrix}\phi_1 \\ \phi_2\end{bmatrix} \in \mathcal{C}^2$, ϕ_1, $\phi_2 \in \mathcal{C}$ as

$$
\begin{aligned}
k(\phi) = V_I\Bigg(&K_{xi}I_{\text{ref}} - \frac{T_{iG\max}}{V_I}\varphi\big(\phi_1(-\tau_g) + G_{\text{ref}}\big) \\
&+ \frac{K_{xgi}}{\rho}\phi_1^2(0) + \frac{K_{xgi}G_{\text{ref}}}{\rho}\phi_1(0)\Bigg) - v_{\text{ref}}
\end{aligned} \tag{30}
$$

with $\rho = 2 \cdot 10^{-5}$. By choosing $u(t) = k(x_t)$, the related closed-loop system is described by (see (29), (30)):

$$
\begin{aligned}
\dot{x}_1(t) &= -K_{xgi}\Big(x_1(t)x_2(t) + I_{\text{ref}}x_1(t) + G_{\text{ref}}x_2(t)\Big) \\
\dot{x}_2(t) &= -K_{xi}x_2(t) + \frac{K_{xgi}}{\rho}x_1^2(t) + \frac{K_{xgi}G_{\text{ref}}}{\rho}x_1(t) \\
x(\tau) &= x_0(\tau), \quad \tau \in [-\tau_g, 0].
\end{aligned} \tag{31}
$$

Notice that, system (31) is in the form (1) where f is the map from $\mathcal{C}^2$ to $\mathbb{R}^2$.

Remark 1: We highlight that, a discretized version of the static state feedback controller provided in (30), ensuring the semi–global practical stability of the related sampled–data glucose–insulin system, has been proposed in [3]. Here, we prove that the same static state feedback applied in continuous–time basis ensures the 0–GALES property of the related continuous–time glucose–insulin system (see (31)).

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Let $P, Q \in \mathbb{R}^{2\times 2}$ be two symmetric positive definite matrices, defined as follows:

$$P = \begin{bmatrix} p_1 & 0 \\ 0 & \rho p_1 \end{bmatrix}, \quad Q = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix},$$

where p_1, q_1, q_2, are positive reals such that $q_1 < 2p_1 K_{xgi} I_{\rm ref}$, $q_2 < 2\rho p_1 K_{xi}$. Let $V : \mathcal{C}^2 \to \mathbb{R}^+$ be the functional defined, for any $\phi \in \mathcal{C}^2$, as

$$V(\phi) = \phi(0)^T P \phi(0) + \int_{-\tau_g}^{0} \phi(\tau)^T Q \phi(\tau)\, d\tau.$$

Let

$$\alpha_1 = \lambda_{\min}(P),\ \alpha_3 = \min\{2p_1 K_{xgi} I_{\rm ref} - q_1,\ 2\rho p_1 K_{xi} - q_2\}. \tag{32}$$

Let α_2 be the function defined, for any $s \in \mathbb{R}^+$, as

$$\alpha_2(s) = \lambda_{\max}(P) + \tau_g \lambda_{\max}(Q). \tag{33}$$

Notice that, the functional V is Lipschitz on bounded sets of $\mathcal{C}^2$ and satisfies the first two inequalities in (9), with the positive real α_1 and the function α_2 in (32), (33). The third inequality in (9) is satisfied (with respect to the system described by (31)), by choosing, for instance, α_3 as in (32). Then, from Theorem 1 the closed–loop system described by (31) is 0–GALES. Simulations have been performed with the following set of parameters allowing to describe an average type 2 diabetic patient (see [17]): $G_b = 8.45$, $I_b = 47.85$, $T_{iG\max} = 1.695$, $\gamma = 15.92$, $G^* = 9$, $\tau_g = 6.5$, $V_G = 0.18$, $K_{xi} = 3.8 \times 10^{-2}$, $T_{gh} = 0.0023$, $V_I = 0.25$, $K_{xgi} = 3.15 \times 10^{-5}$. The initial condition of the system has been chosen equal to $x_0 = \begin{bmatrix} 3.45 \\ -33.2787 \end{bmatrix}$, $\tau \in [-\tau_g, 0]$. In Fig. 1 the evolution of the state variables $x_1(t)$, $x_2(t)$ are reported. Simulations fully validate the results.

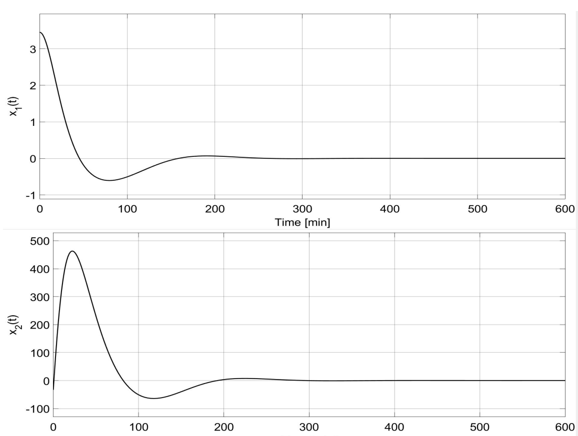

Fig. 1. Evolution of the state variables $x_1(t)$, $x_2(t)$

V. Conclusions

In this paper, a Lyapunov–Krasovskii characterization of the GALES property for systems described by RFDEs is provided. Point–wise dissipation rates are used. In order to show the ease of use of the provided tool, an example, concerning the glucose–insulin regulatory system, has been presented. Highly motivated by [6], future investigations may concern the use of the GALES notion for the stability analysis of nonlinear sampled–data time–delay systems (see [4]).

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